Stars and Bars

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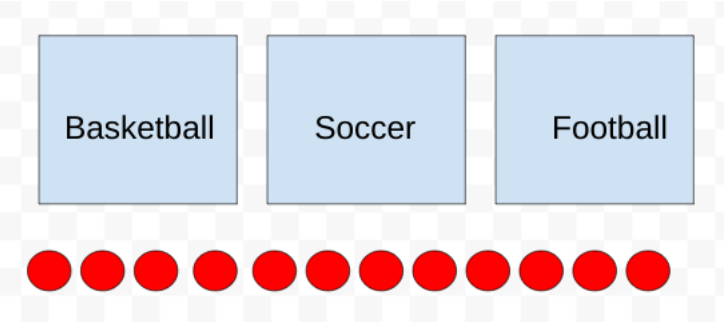
1 Basic Counting

There are first a few definitions we must declare before moving on
Combinations: the number of ways to choose p objects out of n objects where order does not matter is $\boxed{\dfrac{n!}{(p!)(n-p)!}}$ choosing p objects out of n is also written as $\boxed{\binom{n}{p}}$
Permutations: the number of ways to choose p objects out of n objects and arrange them in a line is $\boxed{\dfrac{n!}{(n-p)!}}$ Order matters!
Problem: How many ways to to choose 1 president, 3 vice presidents, and 1 secretary from 10 AYMC members?
solution: Note that we don't care about the order of vice presidents. The sets (Tina, Eric, Albert) and (Albert, Tina, Eric) are equivalent. After we choose 1 president and 3 vice presidents, there are 6 people to choose from to be secretary $10\cdot \binom{9}{3} \cdot 6 = \boxed{5040}$
Problem: There are 3 sports teams: basketball, soccer, and football. How

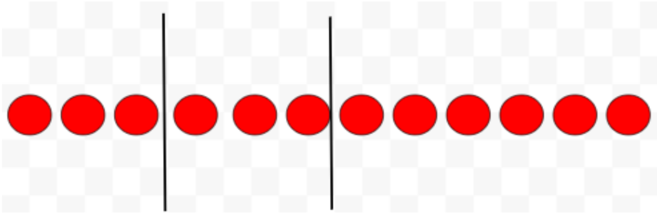
many ways can 12 people join the teams if people are distinguishable? (This means that we can tell people apart. Alan and Darwin on the basketball team is different from Alan and Daniel on the basketball team.)Leave your answer as an exponent. *Solution:* Each person has 3 teams to choose from. There are 12 people, so the answer is $\boxed{3^{12}}$

2 Stars and Bars or balls and urns

Problem: The 3 sports teams are still basketball, soccer, and football. How many ways can 12 people join the teams if people are indistinguishable? (This means that we can NOT tell people apart! Alan and Darwin on the basketball team is "the same" as Alan and Daniel on the basketball team, because 2 people join the basketball team in both scenarios. We care about the number of people who join each team, not who joins the teams.) *Solution* The teams are different, but not the people.



The red dots represent people and the blue boxes represent teams. Without loss of generality, choose the basketball team first, then the soccer team, then the football team. 2 dividers will divide the teams.



This represents that 3 people join the basketball team, 3 people join the soccer team, and 6 people join the football team.
It is also possible for no people to join a team or all people to join a team. What would that look like? How many ways are there to order 12 dots and 2 lines in the diagram? $\binom{14}{2} = \boxed{91}$

Definition 2.1 (Stars and Bars) *The number of ways to assign p indistinguishable items to n distinguishable items is* $\boxed{\frac{(n+p-1)!}{(p-1)!n!}}$

Why is it $n + p - 1$? Like in the example, there are 3 teams but 2 dividers to divide the 3 teams. In general, there are $n - 1$ dividers to divide n distinguishable groups.

3 Examples

Stars and bars can be applied to many scenarios. Some of them are very surprising!

- How many ways are there to give 6 jellybeans to 5 kids?
  There are 4 dividers for 5 kids. $\binom{10}{4} = \boxed{210}$
- How many ways to give 6 jellybeans to 3 kids if each kid had to receive at least 1 jellybean?

First, give each kid 1 jellybean. There are 3 jellybeans left, and 3 kids, which means 2 dividers. $\binom{5}{2}$=$\boxed{10}$

- How many ways can you distribute 3 red lollipops and 2 yellow lollipops to 6 kids?
  Like in the previous problem, we take care of the restriction 1st. First, we order the colored lollipops. $\binom{5}{2}$ Now, we can use stars and bars. The lollipops are stars and kids are bars. $\binom{10}{5}$ The answer is $\binom{5}{2} \cdot \binom{10}{5} =$ $\boxed{2520}$

- How many integer solutions are there to the equation $x+y+z=8$ where x,y, and z are positive?
  What are the stars and what are the bars? There are 2 bars and 8 stars, but we have to distribute 1 star to 3 sections. $\binom{7}{2} = \boxed{21}$

- What if x,y, and z are non-negative? (They can be 0)
  This is stars and bars without restrictions. $\binom{10}{2} = \boxed{45}$

- What if they are all greater than -3?
  First, we add 3 to each number to make them positive. In this case, if it is 0 in the stars and bars, it is actually -3. $x+y+z=17$ So stars and bars $\binom{19}{2} = \boxed{171}$

- How many terms are in the polynomial $(x+y+z)^9$?
  When you expand a polynomial, the sum of the exponents in each term is the degree of the polynomial. In $(x+y+z)^9$, x^9, xy^5z^3, and x^7z^2 are terms. 9, $1+5+3$, and $7+2$ all $=9$. It's like stars and bars where we distribute 9 exponents to 3 variables! $\binom{11}{9} = \boxed{55}$

4 Practice Competition Problems

- (this is from AMC 8) Alice has 24 apples. In how many ways can she share them with Becky and Chris so that each of the three people has at least two apples?
- How many ordered n-tuples of nonnegative integers $x_1, x_2, x_3, x_4 \dots x_n$ are there so that $x_1 + x_2 + x_3 + \cdots + x_n = k$?
- How many degree 6 polynomials $f(x)$ with positive integer coefficients are there such that $f(1) = 30$ and $f(-1) = 12$?
- When $(x+y+z)^{100}$ is expanded out, how many distinct terms are there? For example, $x^{24}y^{32}z^{44}$ is distinct from $x^{24}y^{31}z^{45}$.
- (2001, 19) Pat wants to buy four donuts from an ample supply of three types of donuts: glazed, chocolate, and powdered. How many different selections are possible?
- (2003, 21) Pat is to select six cookies from a tray containing only chocolate chip, oatmeal, and peanut butter cookies. There are at least six of each of these three kinds of cookies on the tray. How many different assortments of six cookies can be selected?
- (2018, 11) When 7 fair standard 6-sided dice are thrown, the probability that the sum of the numbers on the top faces is 10 can be written as

  $$\frac{n}{6^7},$$

  where n is a positive integer. What is n?
- (2016, 10) How many rearrangements of *abcd* are there in which no two adjacent letters are also adjacent letters in the alphabet? For example, no such rearrangements could include either *ab* or *ba*.

- (2017, 17) Call a positive integer monotonous if it is a one-digit number or its digits, when read from left to right, form either a strictly increasing or a strictly decreasing sequence. For example, 3, 23578, and 987620 are monotonous, but 88, 7434, and 23557 are not. How many monotonous positive integers are there?

- (2011, 16) Each vertex of convex pentagon $ABCDE$ is to be assigned a color. There are 6 colors to choose from, and the ends of each diagonal must have different colors. How many different colorings are possible?

5 Solutions

- (this is from AMC 8) Alice has 24 apples. In how many ways can she share them with Becky and Chris so that each of the three people has at least two apples?
  Solution: This is Stars and Bars. First, give everyone 2 apples. There are 18 apples left. 18 apples are 18 stars and 3 people are 2 bars. $\binom{20}{2} = \boxed{190}$

- How many ordered n-tuples of nonnegative integers $x_1, x_2, x_3, x_4 \ldots x_n$ are there so that $x_1 + x_2 + x_3 + \cdots + x_n = k$?
  Solution: This is stars and bars with n distinguishable numbers and k stars. $\binom{(k+n-1)}{(k)} = \boxed{\dfrac{(k+n+1)!}{k!(n-1)!}}$

- How many degree 6 polynomials $f(x)$ with positive integer coefficients are there such that $f(1) = 30$ and $f(-1) = 12$?
  Solution: $f(x) = ax^6 + bx^5 + cx^4 + dx^3 + ex^2 + fx + g$
  $f(1) = a+b+c+d+e+f+g = 30$ $f(-1) = a-b+c-d+e-f+g = 12$
  We can't stars and bars f(1) first, because we have to make sure f(-1) is

12. We add the equations first:
$42 = 2a + 2c + 2e + 2g$ $a + c + e + g = 21$
They can't be 0 because positive integer coefficients. 17 stars left and 3 bars for 4 letters. $\binom{(20}{3)=\frac{20\cdot 19\cdot 18}{3\cdot 2\cdot 1}=570}$
We subtract the equations next:
$2b + 2d + 2f = 18$ $b + d + f = 9$
Similarly, there are 6 stars and 2 bars for 3 letters. $(8c2) = 28$
Ultimately, it is $570 \cdot 28 = \boxed{15960}$

- When $(x + y + z)^{100}$ is expanded out, how many distinct terms are there? For example, $x^{24}y^{32}z^{44}$ is distinct from $x^{24}y^{31}z^{45}$.
  Solution: The exponents of the terms sum to 100. This is like distributing 100 stars to 3 variables. Some variables have a power of 0, in which they equal 1 and won't "show up". Using stars and bars, this is $\binom{100}{2} = \boxed{4950}$

- (2001, 19) Pat wants to buy four donuts from an ample supply of three types of donuts: glazed, chocolate, and powdered. How many different selections are possible?
  Solution: The 4 donuts are the stars and the types are the bars. There are 3 types so there are 2 bars. $\binom{6}{2}=\boxed{15}$

- (2003, 21) Pat is to select six cookies from a tray containing only chocolate chip, oatmeal, and peanut butter cookies. There are at least six of each of these three kinds of cookies on the tray. How many different assortments of six cookies can be selected?
  Solution: The 6 cookies are the stars and the types are the bars. There are 3 types so there are 2 bars. $\binom{8}{2}=\boxed{28}$

- (2018, 11) When 7 fair standard 6-sided dice are thrown, the probability that the sum of the numbers on the top faces is 10 can be written as

$$\frac{n}{6^7},$$

  where n is a positive integer. What is n?
  Solution: The minimum value a dice can roll is 1. If there are 6 1s rolled, there is 1 4. The number of ways to order this is $\binom{7}{1}$=7.
  If there are 5 1s, there is a 2 and a 3. There are $\frac{7!}{5!}$=42 ways to order this.

- (2016, 10) How many rearrangements of *abcd* are there in which no two adjacent letters are also adjacent letters in the alphabet? For example, no such rearrangements could include either *ab* or *ba*.

- (2017, 17) Call a positive integer monotonous if it is a one-digit number or its digits, when read from left to right, form either a strictly increasing or a strictly decreasing sequence. For example, 3, 23578, and 987620 are monotonous, but 88, 7434, and 23557 are not. How many monotonous positive integers are there?

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