Quadratics

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October 1, 2022

Outline

1. What's a quadratic?
2. Factoring
3. Completing the Square... can lead to the formula :D
4. Quadratic Formula
5. Graphing
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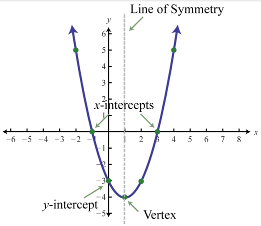
What's a quadratic?

$ax^2 + bx + c$

So... what category do they fall in?

Quadratics when graphed are parabolas, the simplest conic section. Quadratics, when graphed can be expressed as a "curve" function and belongs in the polynomial conics. They have at most 2 roots or zeros. The roots of $Q(x)$, an arbitrary quadratic, are the inputs where the result is 0. Quadratic equations are solved using one of three main strategies: factoring, completing the square and the quadratic formula.



Factoring

Factoring by groups

Take the quadratic $x^2 + 4x + 3$. If we split it up into $x^2 + 3x + x + 3$, we can turn this into 2 groups. The first one, $x^2 + 3x$, can be factored into $x(x + 3)$, and the second one, can be factored into $1(x + 3)$. Our quadratic is now $x(x + 3) + 1(x + 3)$. This can now be factored into $(x + 1)(x + 3)$. If we wanted to find the roots, they would be -1 and -3 because you need at least one zero as a factor in order to make zero.

General factoring

Vieta's Formulas

Vieta's formulas state that for any arbitrary quadratic $ax^2 + bx + c$, the **sum** of the roots is $\frac{-b}{a}$ and the **product** of the roots is $\frac{c}{a}$.

Example

$x^2 - 7x + 12 = 0 \iff$ Notice that -7 is $-3 - 4$, and $(-3) \cdot (-4) = 12$. Thus, $x^2 - 7x + 12 = (x - 3)(x - 4) = 0$. Setting $x - 3 = 0$ and $x - 4 = 0$ give us roots $3, 4$.

Try: Factor $x^2 - 10x + 16$, $x^2 + 9x + 18$, $x^2 - 13x + 42$

Completing the Square... can lead to the formula :D

What is Completing the Square?

Completing the Square is when you add or subtract a value to a quadratic in order to simplify it into a square. Take the quadratic $x^2 + 8x + 12$. Since we are solving for the roots, this is equal to zero. Adding four on both sides gives us $x^2 + 8x + 16 = 4$. Notice that the left hand side, $x^2 + 8x + 16$, is $(x + 4)^2$. Therefore, $(x + 4)^2 = 4$ and $x + 4 = \pm 2$, giving us our two roots, -2 and -6.

Notice that if the middle term had been another number like 4, we would've had to subtract 8 instead to make a perfect square. The eventual expression that will be squared depends on the first and middle term. Just remember that for any $(ax + b)^2$, the first term's coefficient will be a^2 and the middle coefficient will be $2ab$. Using this information, you can figure out what you need to add or subtract in order to Complete the Square.

Vertex Form

Vertex Form

The Vertex Form of a quadratic is $a(x-h)^2+k$, where a, h, and k are constants.

When completing the square, we can keep everything on one side to obtain **Vertex Form**. For example, the vertex form of $x^2+8x+12$ would be $(x+4)^2-4$. Vertex form is very useful because it makes graphing easier and makes finding the minimum/maximum trivial.

$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} + \frac{c}{a} = \frac{b^2}{4a^2}$

$$a(x^2) + b(x) + c = 0 \iff x^2 + (\frac{b}{a})x + \frac{c}{a} = 0$$

$$x^2 + (\frac{b}{a})x + \frac{c}{a} = 0 \iff x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{b^2}{4a^2} - \frac{c}{a}$$

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{b^2}{4a^2} - \frac{c}{a} \iff (x + \frac{b}{2a}) = \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Seems familiar...

Quadratic Formula

Back to $ax^2 + bx + c$

Turns out that indeed, through the method shown, the solutions to a quadratic of the form $ax^2 + bx + c$ are:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Note the plus minus symbol, indicating 2 roots. Also notice the mess under the radical. That is our **discriminant**.

Graphing

Graphing Quadratics

A quadratic will always graph to a parabola, and based on its x^2 coefficient, a, it will open up or down. If a is negative, it will open downwards, and if it is positive, it will open upwards. This is because x^2 will eventually become larger and larger, and will go up or down based on if a is negative or positive. In order to graph a parabola, you always need at least three points. Each parabola will have a **Vertex**. The vertex is the minimum or maximum point of the parabola. If the parabola opens upwards, then it will have a minimum, and if it opens downwards, it will have a maximum. The x value for the vertex will always be **-b/2a**, which you can plug in to find the y value.

Practice Problems

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- For certain real numbers a, b, and c, the polynomial

  $$g(x) = x^3 + ax^2 + x + 10$$

  has three distinct roots, and each root of $g(x)$ is also a root of the polynomial

  $$f(x) = x^4 + x^3 + bx^2 + 100x + c.$$

  What is $f(1)$?
  (A) -9009 **(B)** -8008 **(C)** -7007 **(D)** -6006 **(E)** -5005

more problems!!

- How many ordered pairs of positive integers (b, c) exist where both $x^2 + bx + c = 0$ and $x^2 + cx + b = 0$ do not have distinct, real solutions?
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- The real numbers c, b, a form an arithmetic sequence with $a \geq b \geq c \geq 0$. The quadratic $ax^2 + bx + c$ has exactly one root. What is this root?
  (A) $-7 - 4\sqrt{3}$ **(B)** $-2 - \sqrt{3}$ **(C)** -1 **(D)** $-2 + \sqrt{3}$ **(E)** $-7 + 4\sqrt{3}$

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it happens that $c = \frac{b^2}{4a}$, then the graph of $y = f(x)$ will certainly:
(A) have a maximum (B) have a minimum
(C) be tangent to the x − axis (D) be tangent to the y − axis
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- Let f be a function for which $f\left(\frac{x}{3}\right) = x^2 + x + 1$. Find the sum of all values of z for which $f(3z) = 7$.

(A) $-1/3$ (B) $-1/9$ (C) 0 (D) $5/9$ (E) $5/3$