Triangles, aka Geometry

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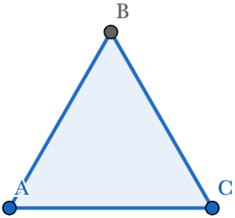
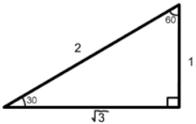
Contents

Triangles and their properties

So what are triangles?

- In case you didn't know (I hope you did), a 2-Dimensional shape is a triangle if and only if it has 3 sides.
- The triangle is the simplest possible polygon within Euclidean Geometry.
- Triangles can be separated into 6 different categories, 3 by their angles, and 3 by their side lengths.





Categories of Triangles

- 1. Acute Triangles
- 2. Obtuse Triangles - There is exactly 1 obtuse angle (which means angle whose degree is larger than 90)
- 3. Right Triangles - There is exactly 1 right angle made by 2 sides (question: Why can't you have 2 right angles within a triangle?)
- 4. Equilateral Triangles - All sides are of equal length
- 5. Isosceles Triangles - At least 2 sides are equal in length
- 6. Scalene Triangles - All sides are of different lengths

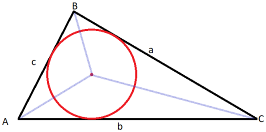
Triangle Areas

Area Formulas

Note: A refers to the area of a triangle

Important formulas!

Base times Height: $\frac{bh}{2} = A$
Incenter, r, and semiperimeter, s: $rs = A$
Heron's formula: $\sqrt{(s)(s-a)(s-b)(s-c)} = A$
Where R is the circumradius: $\frac{abc}{4R} = A$
If you know trig: $\frac{1}{2} \cdot ab \sin c = A$ See if you can see why the 2nd area formula is true with the diagram!



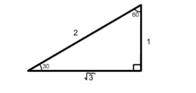
Special Triangles

30-60-90 Triangles and 45-45-90 Triangles

- 30-60-90 Triangles

Definitions

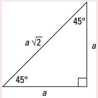
1. Referred to as 30-60-90 triangles because those are the measures of the angles in the triangle.
2. The hypotenuse has a $2 : 1$ ratio with the smaller leg.
3. The larger leg has a $\sqrt{3} : 1$ ratio with the smaller leg.



- 45-45-90 Triangles

Important Facts!

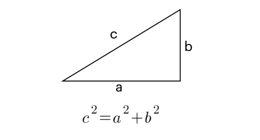
1. Referred to as 45-45-90 triangles because those are the measures of the angles in the triangle.
2. The two legs of the triangle are of equal lengths
3. The ratio between hypotenuse and leg is $\sqrt{2} : 2$

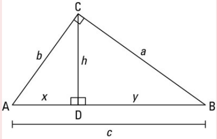


Right Angled Triangles

Theorems

- Pythagorean Theorem: Relates the 2 legs with the hypotenuse. Given legs of length a and b, the length of the hypotenuse is $\sqrt{a^2 + b^2}$. In other words:



- Geometric Mean theorem: 

- Essentially, the theorem states $\sqrt{xy} = h$. Try to see why!
- Practice problem: In quadrilateral $ABCD$, $\angle B$ is a right angle, diagonal $\overline{AC}$ is perpendicular to $\overline{CD}$, $AB = 18$, $BC = 21$, and $CD = 14$. Find the perimeter of $ABCD$.

More special triangles

- EQUILATERAL TRIANGLES

Important Facts!

1. Definition: An equilateral triangle is a triangle in which all three sides have the same length. Note that the angles in equilateral triangle are all 60°

Idea

Every equilateral triangle can be split into 2 identical 30-60-90 triangles.

- ISOSCELES TRIANGLES

Important facts!

1. Definition: Triangles with at least 2 sides of equal length are Isosceles.
2. The two base angles of any isosceles triangle are equal in degree.

Idea

One strategy is dropping the altitude from the vertex to the base of the isosceles triangle. This creates a perpendicular, whose 90° angle can be taken advantage of.

not unique, aka SIMILAR/CONGRUENT TRIANGLES

Similarity theorems

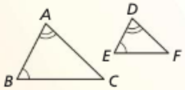
Important theorems!

1. SSS: When all the corresponding sides are in an equal ratio
2. AA: When all the corresponding angles are in an equal ratio
3. SAS: When 2 corresponding sides are in an equal ratio and the angle between is equal
4. ASA: When the angles to the right and left of a side are equal
5. WARNING: SAA IS NOT A SIMILARITY THEOREM!

Concept Summary

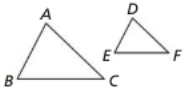
Triangle Similarity Theorems

AA Similarity Theorem



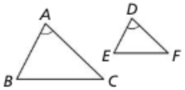
If $\angle A \cong \angle D$ and $\angle B \cong \angle E$, then $\triangle ABC \sim \triangle DEF$.

SSS Similarity Theorem



If $\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}$, then $\triangle ABC \sim \triangle DEF$.

SAS Similarity Theorem



If $\angle A \cong \angle D$ and $\frac{AB}{DE} = \frac{AC}{DF}$, then $\triangle ABC \sim \triangle DEF$.

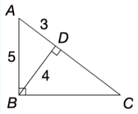
Similarity Properties

Properties

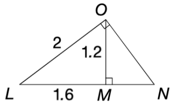
1. All the angles are the same
2. All corresponding sides are in the same ratio
3. The ratio of the area is the ratio between the side lengths squared
4. The other triangle can be created by a dilation of the other triangle
5. These facts come up quite often in various different math contests so ALWAYS LOOK FOR SIMILAR/CONGRUENT TRIANGLES

Practice problem:

Find BC and DC given $AD = 3$, $BD = 4$, and $AB = 5$.



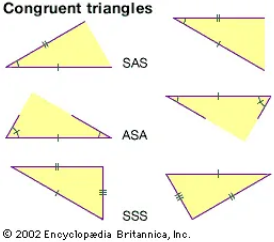
Find ON and MN.



Congruence theorems

Important theorems!

1. SSS: When both triangles have congruent corresponding sides
2. SAS: When both triangles have 2 corresponding sides that are congruent and the angle between the 2 sides is also equal
3. ASA: When both triangles have 2 pairs of congruent angles and the side between the 2 angles is also equal
4. AAS: When both triangles have 2 pairs of congruent angles and the sides not between the 2 angles are equal
5. HL: When 2 right-angled triangles have congruent hypotenuses and 2 congruent legs



Triangle Centers

Circumcenter

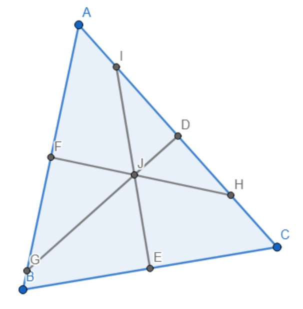
Definition

Perpendicular Bisector: a perpendicular line that divides a segment into 2 equal parts.

- A triangle has 3 perpendicular bisectors, one for each side.

Definition

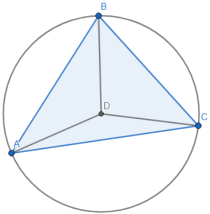
The three perpendicular bisectors intersect at 1 point, called the circumcenter.



Circumcenter

Properties

The distance from the circumcenter to the three verticies is the same. It is the circumradius. If we circumscribe a circle around a triangle, the circumcenter is the center of the circumcircle.



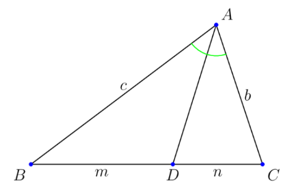
Incenter

Definition

Angle bisectors split an angle into two equal angles.

Theorem

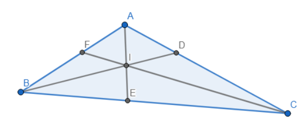
The angle bisector theorem states that given triangle $\triangle ABC$ and angle bisector AD, where D is on side BC, then $\frac{c}{m} = \frac{b}{n}$. It follows that $\frac{c}{b} = \frac{m}{n}$. Likewise, the converse of this theorem holds as well. $AD^2 = b \cdot c - m \cdot n$



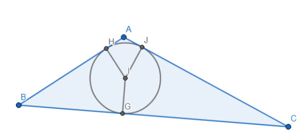
Incenter

- A triangle has 3 angle bisectors, 1 for each vertex.

Definition

The three angle bisectors intersect at the incenter. 

Property

The incenter is the center of a circle inscribed in a triangle. This circle is tangent to all sides of the triangle. 

Centroid

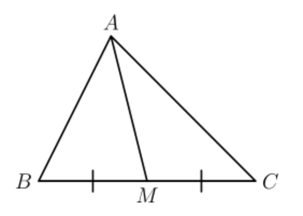
Definition

A median is a line segment from a vertex to the midpoint of the opposite side

- Guess how many medians a triangle has?

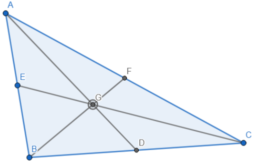
Formula

length of median formula: $2AM^2 = AB^2 + AC^2 - BM^2 - MC^2$



Centroid

- The medians of a triangle intersect at the centroid.
- The distance from the centroid to the vertex is twice the distance from the centroid to the corresponding side. In other words, AG=2GD
- The medians divide the triangle into 6 small triangles of equal area.

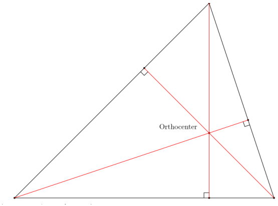


Orthocenter

Definition

An altitude, or height, is a line segment from a vertex perpendicular to the opposite side.

- The three altitudes of a triangle intersect at the orthocenter.



- The right angles of the heights are useful.

Basic intro to Trigonometry

What is trig?

Definition

Trigonometry is the study of relations between the side lengths and angles of triangles through the trigonometric functions.

Trigonometric Functions

- Sine-Opposite over Hypotenuse
- Cosine-Adjacent over Hypotenuse
- Tangent-Opposite over Adjacent

One way to memorize this is with the acronym SOHCAHTOA. Sin is O/H, Cos is A/H, Tan is O/A.

Not really nessecary in AMC 10, but knowledge of trigonometry can speed things up quickly.

Trigonometry Cont.

Important Laws

In a triangle ABC, the following holds:
Law of Cosines: $c^2 = a^2 + b^2 - 2ab\cos(C)$ What does this look like?
Law of Sines: $\frac{A}{\sin(A)} = \frac{B}{\sin(B)} = \frac{C}{\sin(C)}$ where $\sin(x)$ refers to the angle at point x

However, again, since this is AMC 10, I will not be going into the many Trigonometric Identities. You might need those for AMC 12 though :)

Practice Questions

AMC 10

- (2019 Problem 10)In a given plane, points A and B are 10 units apart. How many points C are there in the plane such that the perimeter of triangle ABC is 50 units and the area of triangle ABC is 100 square units?
- (2019 Problem 15)Right triangles T_1 and T_2 have areas 1 and 2, respectively. A side of T_1 is congruent to a side of T_2, and a different side of T_1 is congruent to a different side of T_2. What is the square of the product of the other (third) sides of T_1 and T_2?
- (2019 Problem 16)In triangle ABC with a right angle at C, point D lies in the interior of $\overline{AB}$ and point E lies in the interior of $\overline{BC}$ so that $AC = CD$, $DE = EB$, and the ratio $AC : DE = 4 : 3$. What is the ratio $AD : DB$?
- (2018 Problem 16) Right triangle ABC has leg lengths $AB = 20$ and $BC = 21$. Including $\overline{AB}$ and $\overline{BC}$, how many line segments with integer length can be drawn from vertex B to a point on hypotenuse $\overline{AC}$?
- (2021 Problem 15) Isosceles triangle ABC has $AB = AC = 3\sqrt{6}$, and a circle with radius $5\sqrt{2}$ is tangent to line AB at B and to line AC at C. What is the area of the circle that passes through vertices A, B, and C?

AMC 10 continued

- (2017 Problem 19) Let ABC be an equilateral triangle. Extend side $\overline{AB}$ beyond B to a point B' so that $BB' = 3 \cdot AB$. Similarly, extend side $\overline{BC}$ beyond C to a point C' so that $CC' = 3 \cdot BC$, and extend side $\overline{CA}$ beyond A to a point A' so that $AA' = 3 \cdot CA$. What is the ratio of the area of $\triangle A'B'C'$ to the area of $\triangle ABC$?
- (2017 Problem 21) In $\triangle ABC$, $AB = 6$, $AC = 8$, $BC = 10$, and D is the midpoint of $\overline{BC}$. What is the sum of the radii of the circles inscribed in $\triangle ADB$ and $\triangle ADC$?
- (2017 Problem 21) A square with side length x is inscribed in a right triangle with sides of length 3, 4, and 5 so that one vertex of the square coincides with the right-angle vertex of the triangle. A square with side length y is inscribed
- (2018 Problem 23) Farmer Pythagoras has a field in the shape of a right triangle. The right triangle's legs have lengths 3 and 4 units. In the corner where those sides meet at a right angle, he leaves a small unplanted square S so that from the air it looks like the right angle symbol. The rest of the field is planted. The shortest distance from S to the hypotenuse is 2 units. What fraction of the field is planted?
- (2018 Problem 24) Triangle ABC with $AB = 50$ and $AC = 10$ has area 120. Let D be the midpoint of $\overline{AB}$, and let E be the midpoint of $\overline{AC}$. The angle bisector of $\angle BAC$ intersects $\overline{DE}$ and $\overline{BC}$ at F and G, respectively. What is the area of quadrilateral $FDBG$?

AMC10 geometry

- (2017,10) Joy has 30 thin rods, one each of every integer length from 1 cm through 30 cm. She places the rods with lengths 3 cm, 7 cm, and 15 cm on a table. She then wants to choose a fourth rod that she can put with these three to form a quadrilateral with positive area. How many of the remaining rods can she choose as the fourth rod?
- (2018,13) A paper triangle with sides of lengths $3, 4,$ and 5 inches, as shown, is folded so that point A falls on point B. $AB = 5$ and $AC = 4$ What is the length in inches of the crease?
- (2018,15) Two circles of radius 5 are externally tangent to each other and are internally tangent to a circle of radius 13 at points A and B. The distance AB can be written in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. What is $m + n$?
- (2017,21) A square with side length x is inscribed in a right triangle with sides of length 3, 4, and 5 so that one vertex of the square coincides with the right-angle vertex of the triangle. A square with side length y is inscribed in another right triangle with sides of length 3, 4, and 5 so that one side of the square lies on the hypotenuse of the triangle. What is $\dfrac{x}{y}$?
- (2017,22) Sides $\overline{AB}$ and $\overline{AC}$ of equilateral triangle ABC are tangent to a circle at points B and C respectively. What fraction of the area of $\triangle ABC$ lies outside the circle?

AIME

- (2019,1)Two different points, C and D, lie on the same side of line AB so that $\triangle ABC$ and $\triangle BAD$ are congruent with $AB = 9$, $BC = AD = 10$, and $CA = DB = 17$. The intersection of these two triangular regions has area $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.
- (2022,3)In isosceles trapezoid $ABCD$, parallel bases $\overline{AB}$ and $\overline{CD}$ have lengths 500 and 650, respectively, and $AD = BC = 333$. The angle bisectors of $\angle A$ and $\angle D$ meet at P, and the angle bisectors of $\angle B$ and $\angle C$ meet at Q. Find PQ.
- (2011, 4) In triangle ABC, $AB = 125$, $AC = 117$ and $BC = 120$. The angle bisector of angle A intersects $\overline{BC}$ at point L, and the angle bisector of angle B intersects $\overline{AC}$ at point K. Let M and N be the feet of the perpendiculars from C to $\overline{BK}$ and $\overline{AL}$, respectively. Find MN.
- (2011, 4)In triangle ABC, $AB = 20$ and $AC = 11$. The angle bisector of $\angle A$ intersects BC at point D, and point M is the midpoint of AD. Let P be the point of the intersection of AC and BM. The ratio of CP to PA can be expressed in the form $\dfrac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.
- (2015, 4)Point B lies on line segment $\overline{AC}$ with $AB = 16$ and $BC = 4$. Points D and E lie on the same side of line AC forming equilateral triangles $\triangle ABD$ and $\triangle BCE$. Let M be the midpoint of $\overline{AE}$, and N be the midpoint of $\overline{CD}$. The area of $\triangle BMN$ is x. Find x^2.

AIME continued

- (2016, 6) In $\triangle ABC$ let I be the center of the inscribed circle, and let the bisector of $\angle ACB$ intersect AB at L. The line through C and L intersects the circumscribed circle of $\triangle ABC$ at the two points C and D. If $LI = 2$ and $LD = 3$, then $IC = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.
- (2019, 7)Triangle ABC has side lengths $AB = 120, BC = 220$, and $AC = 180$. Lines ℓ_A, ℓ_B, and ℓ_C are drawn parallel to $\overline{BC}, \overline{AC}$, and $\overline{AB}$, respectively, such that the intersections of ℓ_A, ℓ_B, and ℓ_C with the interior of $\triangle ABC$ are segments of lengths $55, 45$, and 15, respectively. Find the perimeter of the triangle whose sides lie on lines ℓ_A, ℓ_B, and ℓ_C.